Contents

1 Preliminaries

1.1 Definitions

Consider $\mathbb{Z}_c \overset{\text{def}}{=\!=} \{0, 1, 2, 3, \ldots, c-1\}$, the set of integers modulo c, and consider the Cartesian product $\mathbb{Z}_c^b$ of b-tuples with entries in $\mathbb{Z}_c$. This alphabet, taken as b-strings in base c, will give us the vertices of a family of graphs. For an element $x \in \mathbb{Z}_c^b$, we will denote its entries with $x = (x_1, \ldots, x_b)$, with all of the entries $x_i \in \mathbb{Z}_c$ and an implied indexing set $i \in [b] \overset{\text{def}}{=\!=} \{1, 2, \ldots, b\}$. We will denote by $\overline{0} \in V$ the zero string, the string which takes on a zero in every index. We will now define the basis for a nontransitive relation on these strings by comparing their entries indexwise.

Definition 1.1 (Skeletons, Hamming Distance)**.** Let $x, y \in \mathbb{Z}_c^b$ be two strings and define their *skeleton* to be the collection of indices $\mathrm{Sk}\{x, y\} \subseteq [b]$ at which their entries differ. In the set notation,

$$\mathrm{Sk}\{x, y\} \overset{\text{def}}{=\!=} \{i \in [b] : x_i \neq y_i\}.$$

For any two strings $x, y \in \mathbb{Z}_c^b$ define their *Hamming distance* $\boldsymbol{d}(x, y)$, to be the number of indices at which their entries differ. That is,

$$\boldsymbol{d}(x, y) \overset{\text{def}}{=\!=} \#\mathrm{Sk}\{x, y\}.$$

Example 1.1.1 (Some Hamming Distance Relations). The following table uses the analogy of color to enable a visual intuition. Note that the number of nonzero entries is an easy way to spot distance from zero.

Point ($\mathbb{Z}_3$)	**Color**	**d=1**	**d=2**	**Point ($\mathbb{Z}_3$)**	**Color**	**d=1**	**d=2**
$00 \equiv 0$	RR	1,2,3,6	4,5,7,8	$12 \equiv 5$	GB	2,3,4,8	0,1,6,7
$01 \equiv 1$	RG	0,2,4,7	3,5,6,8	$20 \equiv 6$	BR	0,3,7,8	1,2,4,5
$02 \equiv 2$	RB	0,1,5,8	3,4,6,7	$21 \equiv 7$	BG	1,4,6,8	0,2,3,5
$10 \equiv 3$	GR	0,4,5,6	1,2,7,8	$22 \equiv 8$	BB	2,5,6,7	0,1,3,4
$11 \equiv 4$	GG	1,3,5,7	0,2,6,8				

Using this notion, we are ready to define the family of graphs of interest.

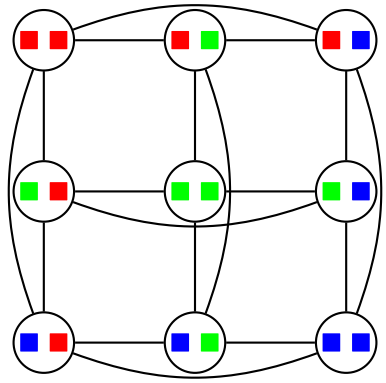
Definition 1.2 (Weakly-General Hamming Graph)**.** For any positive integers b, c and an integer d such that $0 \leqslant d \leqslant b$, define the *weakly-general Hamming graph* $\mathcal{H}_c^b(d)$ to be the graph whose vertices are written as in $\mathbb{Z}_c^b$, with edges given by the set

$$\mathcal{E}_c^b(d) \overset{\text{def}}{=\!=} \{\{x, y\} : x, y \in \mathbb{Z}_c^b, \boldsymbol{d}(x, y) = d\},$$

We say that d is the *Hamming distance* corresponding to the graph $\mathcal{H}_c^b(d)$, or simply the distance in $\mathcal{H}_c^b(d)$.

Remark 1.3. One notes this graph is identified by the parameters b, c, d. Also, note $\mathcal{H}_c^b(1) = H(b, c)$, where $H(b, c)$ denotes a Hamming graph under the standard definition. In some sense, the family of weakly-general Hamming graphs expands upon the standard definition of a Hamming graph to allow for a single choice among all distances. Though, since this particular choice of generalization allows only for the selection of one distance at a time, it is in some sense weaker than other common generalizations of the Hamming graph which do not have this same single-distance restriction. There, a condition of $\boldsymbol{d}(x, y) \leqslant d$ is more typical.

Example 1.3.1 (A Small Picture). In our way of doing things, this is written $\mathcal{H}_3^2(1)$.



Within the family of graphs $\mathcal{H}_c^b(d)$ for all possible b, c, d, our main question and investigations revolve around the enumeration of complete subgraphs as a function of these parameters. We present a standard definition in the context of graphs. If it helps, borrow your intuition from the word clique in social contexts.

Definition 1.4 (Clique, Maximal Clique, Order)**.** A *clique* in a graph G with vertices V and edges E is a subset $\chi \subseteq V$ such that $\forall x, y \in \chi, \{x, y\} \in E$. That is, a clique is a complete subgraph of G, such that every pair of vertices is an edge. A clique χ is said to be *maximal* if there does not exist a larger clique χ' for which $\chi \subset \chi'$. That is, it is maximal if it is not a subclique of a strictly larger clique. We say the size of a clique χ as a set of vertices is its *order*.

Here, we pause to develop some useful notations. Let C denote the set of cliques in $\mathcal{H}_c^b(d)$ and $\hat{C}$ its set of maximal cliques. Let $S \subseteq \mathbb{Z}_c^b$ and choose some integer $\ell \geqslant 0$. We define the family of subsets

$$C_\ell(S) \stackrel{\text{def}}{=\!=} \{\chi \in C : S \subseteq \chi, \#\chi = \ell, \forall x, x' \in \chi, \boldsymbol{d}(x, x') = d\},$$

which gives the set of cliques of order ℓ which contain S in the graph $\mathcal{H}_c^b(d)$. Note $C_\ell(S)$ will be empty if $S \notin C$ or $\ell < \#S$. Define similarly the family of subsets containing maximal cliques with specified order and containing specified subsets of $\mathbb{Z}_c^b$,

$$\hat{C}_\ell(S) \stackrel{\text{def}}{=\!=} \{\chi \in C_\ell(S) : \forall x \in \left(\mathbb{Z}_c^b \backslash \chi\right), \exists x' \in \chi, \boldsymbol{d}(x, x') \neq d\}.$$

For simplicity, we will use some shorthand to allow us to generalize our references to these families $C_\ell(S), \hat{C}_\ell(S)$ of cliques. Whenever our required set S is empty, restricting us none, we omit the parenthetical subset. The set of all cliques of order ℓ is denoted C_ℓ. Similar shorthand can be applied to order, omitting ℓ when we want to consider cliques of any size that contain a specified set.

Definition 1.5 (Clique Polynomial)**.** Any finite graph G has as its clique polynomial

$$C_G(t) = \sum_{\ell \geqslant 0} c_\ell t^\ell,$$

where each coefficient c_ℓ is the number of cliques of order ℓ in the graph G. In our notations for $\mathcal{H}_c^b(d)$, see that $c_\ell = \#C_\ell$, the size of the collection of all ℓ-cliques.

We now introduce without proof a well-known result in Graph theory, which will become useful in context when we have the result that every graph $\mathcal{H}_c^b(d)$ is regular, simplifying greatly the left hand side of the statement of the lemma.

Definition 1.6 (Degree of Vertex, Neighborhood)**.** Let v be a vertex in a graph G. The *degree* of v is the number of vertices x in G such that the edge $\{v, x\}$ exists in G. Letting $N(v)$ be this *open neighborhood*, we write $\deg(v) \stackrel{\text{def}}{=\!=} \#N(v)$.

Lemma 1.7 (Handshaking Lemma)**.** *Let G be a finite graph with vertices V and edge set E. Then,*

$$\sum_{v \in V} \deg(V) = 2\#E.$$

We also invoke an important enumeration theorem from Group theory.

Theorem 1.8 (Orbit-Stabilizer)**.** *Let G be a group which acts on a finite set X. Then, for every $x \in X$, $\#Orb_G(x) = \frac{\#G}{\#Stab_G(x)}$.*

1.2 A Group of Automorphisms

Given the vertices $\mathbb{Z}_c^b$ of an expanded Hamming graph $\mathcal{H}_c^b(d)$, we will work to secure a group of automorphisms of the graph before studying some basic properties of its application to vertices of $\mathcal{H}_c^b(d)$.

For a weakly general Hamming graph $\mathcal{H}_c^b(d)$, its associated group is $S_c \wr S_b$, the wreath product of finite symmetric groups S_c, S_b corresponding to permutations on the underlying c-alphabet $\mathbb{Z}_c$ and the set of indices $[b]$, respectively. We shall denote an element $g \in S_c \wr S_b$ by $g = (\tau_1, \tau_2, \ldots, \tau_b, \sigma)$, where $\sigma \in S_b$ and $\tau_i \in S_c$ for each $i \in [b]$. The operation of $S_c \wr S_b$ is given by $g'g = (\tau_1'\tau_{\sigma'^{-1}(1)}, \tau_2'\tau_{\sigma'^{-1}(2)}, \ldots, \tau_b'\tau_{\sigma'^{-1}(b)}, \sigma'\sigma)$, with elements $g' = (\tau_1', \tau_2', \ldots, \tau_b', \sigma')$ and $g = (\tau_1, \tau_2, \ldots, \tau_b, \sigma)$ in $S_c \wr S_b$.

We want to define an action on $\mathbb{Z}_c^b$ (thus, being able to apply this to its subsets, including cliques) via this product. To do so, recall that the vertex v has b entries, each in $\mathbb{Z}_c$.

Proposition 1.9 (Action on Strings)**.** $S_c \wr S_b \curvearrowright \mathbb{Z}_c^b$

Proof. Given $v \in \mathbb{Z}_c^b$ and $g = (\tau_1, \tau_2, \ldots, \tau_b, \sigma) \in S_c \wr S_b$, let g map v indexwise by

$$g(v)_i \stackrel{\text{def}}{=\!=} \tau_i(v_{\sigma^{-1}(i)}).$$

Let $g = (\tau_1, \tau_2, \ldots, \tau_b, \sigma)$ and $g' = (\tau_1', \tau_2', \ldots, \tau_b', \sigma')$ be elements of the wreath product. Consider the action of g' on $g(v)$. We have $g(v)_{\sigma'^{-1}(i)} = \tau_{\sigma'^{-1}(i)}(v_{\sigma^{-1}(\sigma'^{-1}(i))})$. So, decomposing this action indexwise,

$$\begin{aligned} g'(g(v))_i &= \tau_i'(\tau_{\sigma'^{-1}(i)}(v_{\sigma^{-1}(\sigma'^{-1}(i))})) \\ &= (\tau_i'(\tau_{\sigma'^{-1}(i)}(v_{(\sigma'\sigma)^{-1}(i)}))) \\ &= (g'g)(v)_i. \end{aligned}$$

♣

When we write the action of elements of $S_c \wr S_b$ on subsets of $\mathbb{Z}_c^b$, we take the standard, pointwise definition. We now state and prove one more proposition, intending to calibrate our own sense of freedom in acting upon the vertices of a weakly-general Hamming graph using its associated group.

Proposition 1.10 (Transitive Action on Vertices)**.** *The group $S_c \wr S_b$ acts transitively on $\mathbb{Z}_c^b$.*

Proof. Take any $v \in \mathbb{Z}_c^b$. We will first show $\#\text{Stab}_{S_c \wr S_b}(v) = b!((c-1)!)^b$. For any $\sigma \in S_b$, consider an element $(\tau_1, \tau_2, \ldots, \tau_b, \sigma) \in S_c \wr S_b$. For any index i, in order to ensure that $g(v)_i = v_i$, we must have τ_i be such that $v_i = \tau_i(v_{\sigma^{-1}(i)})$. Thus, τ_i can be any permutation on $\mathbb{Z}_c$ for which this single pairing holds. So, fixing the permutation σ, we find at each index a choice of $(c-1)!$ elements τ_i that will satisfy this.

See that the indices for which we select the permutations $\tau_i \in S_c$ are independent, and two elements of $S_c \wr S_b$ that correspond to distinct permutations $\sigma, \sigma' \in S_b$ are distinct. Therefore, $\#\text{Stab}_{S_c \wr S_b}(v) = b!((c-1)!)^b$. Recall that $\#S_c \wr S_b = b!(c!)^b$. We apply the Orbit-Stabilizer Theorem, and we find

$$\begin{aligned} \#\text{Orb}_{S_c \wr S_b}(v) &= \frac{b!(c!)^b}{b!((c-1)!)^b} \\ &= c^b, \end{aligned}$$

where we observe that this is nothing but $\#\mathbb{Z}_c^b$. ♣

The transitivity of action result on edges also follows from counting the relevant stabilizer subgroup, but does in some sense need a stronger foundation. Here, I just want to give you some intuition for what this is actually doing under the hood.

Remark 1.11. This product of groups is really a row or column permutation we can apply to cliques at a time. The column permutations are simply a reassignment of our alphabet. The structure of $S_c \wr S_b$ says to choose a total of b permutations in S_c (so, on a set of c colors), and then has a permutation on our indices from S_b to also allow us to reindex the result clique. Think of this like giving us the power to, for example, deconstruct all the cliques in a pile and put them back so their second and seventh elements are swapped, but they are otherwise unchanged.

Together, the entire product allows us to start with a clique, change the order of its indices (such that a clique with uniform first index can, for example, also tell us something about a clique where the last index is uniform and the first index has some disagreements between some strings), as well as perform operations on the entries in targeted columns, with what we will soon show is enough surgical precision to preserve the Hamming distance.

Example 1.11.1 (Anywhere to Zero). (Here I will use color to show how to construct the element that gets us from anywhere to the zero. This will help make the group's action more natural, seeing it constructed with so little effort. The symbols are trash, but the group and action are just intuition.)

We will next show that, under the action given in Proposition 1.9, this group is a group of automorphisms of $\mathcal{H}_c^b(d)$. This will allow us to prove that the action of $S_c \wr S_b$ is transitive on the set $\mathcal{E}_c^b(d)$. First, we will consider the effect of this group action on the skeletons associated to vertices under some element of $S_c \wr S_b$.

Lemma 1.12 (Skeleton Permutations). *Let $v, x \in \mathbb{Z}_c^b$ and $g \in S_c \wr S_b$ be given by $g = (\tau_1, \tau_2, \ldots, \tau_b, \sigma)$. Then, $i \in Sk\{v, x\} \iff \sigma(i) \in Sk\{g(v), g(x)\}$.*

Proof. Let $i \in \mathrm{Sk}\{v, x\}$. Then, $v_i \neq x_i$. Thus, $g(v)_{\sigma(i)} = \tau_{\sigma(i)}(v_i) \neq \tau_{\sigma(i)}(x_i) = g(x)_{\sigma(i)}$. If instead we assume $i \notin \mathrm{Sk}\{v, x\}$, then $v_i = x_i$ and we get $g(v)_{\sigma(i)} = \tau_{\sigma(i)}(v_i) = \tau_{\sigma(i)}(x_i) = g(x)_{\sigma(i)}$. ♣

Now, using this lemma, we will see that this action preserves Hamming distances.

Theorem 1.13 ($S_c \wr S_b$ are automorphisms of $\mathcal{H}_c^b(d)$)**.** *Consider the set $\mathbb{Z}_c^b$. Any $g \in S_c \wr S_b$ is such that, for all pairs $x, y \in \mathbb{Z}_c^b$, $\boldsymbol{d}(x, y) = \boldsymbol{d}(g(x), g(y))$.*

Proof. Let $x, y \in \mathbb{Z}_c^b$. It follows immediately from Lemma 1.12 that $\sigma(\mathrm{Sk}\{x, y\}) = \mathrm{Sk}\{g(x), g(y)\}$ and so $\#\mathrm{Sk}\{x, y\} = \#\mathrm{Sk}\{g(x), g(y)\}$, where by the definition of the Hamming distance this is exactly what we wanted to show. ♣

Observation 1.14 (Clique Preservation)**.** *Let us have a clique $\chi \in C_\ell(S)$ and an automorphism $g : S \mapsto T$. Then, $g(\chi) \in C_\ell(T)$.*

Proposition 1.15 (Preservation of Maximality)**.** *Let χ be a maximal clique in $\mathcal{H}_c^b(d)$ and $g \in S_c \wr S_b$. Then, $g(\chi)$ is maximal also.*

Proof. By way of contradiction, we assume there exists a maximal clique $\chi \in \hat{C}$ and automorphism $g \in S_c \wr S_b$ such that $g(\chi) \notin \hat{C}$. This implies there exists a vertex $x \in \mathbb{Z}_c^b \backslash g(\chi)$ nonempty such that $g(\chi) \cup \{x\}$ is a clique.

But, by preservation and bijectivity, the action of g^{-1} on the union $g(\chi) \cup \{x\}$ will give us a clique $\chi \cup g^{-1}(x)$ strictly larger than χ. As such, since we have $\chi \subset \chi \cup g^{-1}(x)$, we see that χ is not maximal. Hence, $\chi \in \hat{C}$ will guarantee $g(\chi) \in \hat{C}$ also for any g. ♣

1.3 Graph Theoretic Properties

The following definition is standard.

Definition 1.16 (Regular Graph)**.** Let G be a graph with vertex set V. We say that G is k-regular if, for all $v \in V$, $\deg(v) = k$.

We will now show that every weakly-general Hamming graph is a regular graph. In doing so, we will define and motivate the use of a piece of notation that will stay with us.

Definition 1.17 (Neighborhood of v)**.** Let $v \in \mathbb{Z}_c^b$. The neighborhood of v, denoted $N(v)$, is the set $N(v) = \{x \in \mathbb{Z}_c^b : \boldsymbol{d}(v, x) = d\}$.

The next theorem involves performing a simple enumeration task on these neighborhoods.

Theorem 1.18 (Weakly-General Hamming Graphs are Regular Graphs)**.** *The weakly-general Hamming graph $\mathcal{H}_c^b(d)$ is $\deg(\mathcal{H})$-regular, where $\deg(\mathcal{H}) = \binom{b}{d}(c-1)^d$*

Proof. Take any $v \in \mathbb{Z}_c^b$, and we will combinatorially prove its degree is $\deg(\mathcal{H})$. To do so, we count the number of choices of $x \in \mathbb{Z}_c^b$ such that $\#\mathrm{Sk}\{v, x\} = d$. Clearly, if this is the case, then there exists exactly d indices at which they differ. These can be chosen in any of $\binom{b}{d}$ ways from the indexing set $[b]$.

And, for each such arrangement, we choose independently at each index $i \in \mathrm{Sk}\{v, x\}$ the entry x_i from the set $\mathbb{Z}_c \backslash v_i$, which contains $c-1$ elements. And, there are $\#\mathrm{Sk}\{v, x\} = d$ indices at which we make such a choice. Elsewhere, we are fixed by indexwise agreement with v. From here, one takes a product to recover the formula $\deg(v) = \binom{b}{d}(c-1)^d$, and notices this holds for any v, which proves the graph is regular of the claimed degree. ♣

Using this fact, the handshaking lemma gives us an immediate simplification, to be stated in the following corollary.

Corollary 1.18.1 (Size of $\mathcal{E}_c^b(d)$)**.** $\#\mathcal{E}_c^b(d) = \frac{c^b}{2}\binom{b}{d}(c-1)^d$.

Proof. Since $\#\mathbb{Z}_c^b = c^b$, we apply $\left(\binom{b}{d}(c-1)^d\right)$-regularity from the theorem to the Handshaking Lemma 1.7. ♣

Remark 1.19. At this stage, we have that the first three coefficients of $C_{\mathcal{H}_c^b(d)}(t)$ are

$$\begin{cases} c_0 = 1 \\ c_1 = c^b \\ c_2 = \frac{c^b}{2}\binom{b}{d}(c-1)^d. \end{cases}$$

2 Automorphisms and Enumeration

2.1 Tools of Local Clique Enumeration

First, we will define some useful notations to use in our description of the global problem. Taking a clique of a certain size, we want to define a notation which motivates our ability to count larger cliques which the smaller clique seems to be embedded in. In some sense, we aim to label a clique in this way and show that cliques with the same labels have exploitable combinatorial properties, up to our choosing smart automorphisms. We will quickly run into a wall, but we need not push this idea very far.

Definition 2.1 (ℓ-Labels)**.** Let $r \in \mathbb{N}$ be such that $S_c \wr S_b$ acts transitively on C_r. Fix $\Upsilon \in C_r$, and let $\ell \in \mathbb{N}$. We define the ℓ-label of order r by

$$\varphi_r(\ell) = \#C_\ell(\Upsilon).$$

The next proposition verifies this is well-defined.

Proposition 2.2 (Weak Local Count Universality)**.** *Given $S_c \wr S_b$ acts transitively on C_r, the label $\varphi_r(\ell)$ is well-defined, taking only one value across all choices of $\Upsilon \in C_r$.*

Proof. Let $\Upsilon, \Upsilon' \in C_r$ be two arbitrary r-cliques. By the assumption of transitivity of the action of the wreath product, there exists some automorphism $g \in S_c \wr S_b$ for which $g(\Upsilon) = \Upsilon'$. The natural map

$$\begin{aligned} g : C_\ell(\Upsilon) &\to C_\ell(\Upsilon') \\ \chi &\mapsto g(\chi) \end{aligned}$$

is a bijection, and so it is clear that $\#C_\ell(\Upsilon) = \#C_\ell(\Upsilon)'$ for all pairs $\Upsilon, \Upsilon' \in C_r$ in which case the result follows by transitivity. ♣

An interesting and necessary side quest enabled by this theorem is the anicipated transitivity result on the set $\mathcal{E}_c^b(d)$.

Theorem 2.3 (Transitive Action on Edges)**.** *Consider a graph $\mathcal{H}_c^b(d)$ with the associated group $S_c \wr S_b$. $S_c \wr S_b \rightharpoondown \mathcal{E}_c^b(d)$ transitively.*

Proof. Let $\{v, x\}$ be an edge, and let us count elements of $S_c \wr S_b$ that stabilize it, toward an application of the Orbit-Stabilizer Theorem. Only the permutations $\sigma \in S_b$ that will fix $\mathrm{Sk}\{v, x\}$ will be present in $\mathrm{Stab}_{S_c \wr S_b}\{v, x\}$. Since the skeleton of an edge is a d-set, one readily verifies that there are $d!(b - d!)$ permutations σ that do this.

Suppose that $i \in \mathrm{Sk}\{v, x\}$ for some such σ. To ensure that $(\tau_1, \ldots, \tau_b, \sigma)$ fixes the edge $\{v, x\}$, we require that $\tau_i(v_{\sigma^{-1}(i)})$ is one of v_i or x_i, and that the other is $\tau_i(x_{\sigma^{-1}(i)})$. Here, we have freedom over the images of $c-2$ elements of $\mathbb{Z}_c$ under τ_i, for each i in a d-set. Technically, we may choose whether $v \mapsto v$ or $v \mapsto x$, but one notes that we must apply the same such choice to all $i \in \mathrm{Sk}\{v, x\}$, so this realization needs only be accounted for by a doubling later. Now, for all indices $i \notin \mathrm{Sk}\{v, x\}$, we require $\tau_i : v_{\sigma^{-1}(i)} = x_{\sigma^{-1}(i)} \mapsto v_i = x_i$. That is, in all $b - d$ such indices, the image under τ_i of one element is determined by its membership in the stabilizer subgroup, so $c - 1$ elements are free.

Doing the accounting and applying the product principle, we have for each σ that there are $2((c - 2)!)^d((c-1)!)^{b-d}$ elements $(\tau_1, \ldots, \tau_b, \sigma) \in \mathrm{Stab}_{S_c \wr S_b}\{v, x\}$. And, since there are $d!(b-d!)$ valid permutations σ for this purpose, we recover $\#\mathrm{Stab}_{S_c \wr S_b}\{v, x\} = 2d!(b-d)!((c-2)!)^d((c-1)!)^{b-d}$. Since we are working with automorphisms of the graph, recall that the orbit of an edge is a subset of the edges. By an arithmetic quotient, we find

$$\begin{aligned} \#\mathrm{Orb}_{S_c \wr S_b}\{v, x\} &= \frac{b!(c!)^b}{2d!(b-d)!((c-2)!)^d((c-1)!)^{b-d}} \\ &= \frac{c^b \binom{b}{d}(c-1)^d}{2} \\ &= \#\mathcal{E}_c^b(d). \end{aligned}$$

♣

The following statement is related, but not a direct corollary. It gives us something stronger than the above, should we need it.

Proposition 2.4 (Clown Lemma)**.** *Let $x, x' \in N(v)$. Then there exists an element $g \in \mathrm{Stab}_{S_c \wr S_b}(v)$ such that $g(x) = x'$.*

Proof. This can be shown via the Orbit-Stabilizer Theorem, keeping track of the size of the subgroup of $\mathrm{Stab}_{S_c \wr S_b}(v)$ that also fixes x and showing that $\deg(\mathcal{H}) = \frac{\#\mathrm{Stab}_{S_c \wr S_b}(v)}{\#\mathrm{Stab}_{S_c \wr S_b}(v) \cap \mathrm{Stab}_{S_c \wr S_b}(x)}$. This holds iff

$$\begin{aligned} \#\mathrm{Stab}_{S_c \wr S_b}(v) \cap \mathrm{Stab}_{S_c \wr S_b}(x) &= \frac{\#\mathrm{Stab}_{S_c \wr S_b}(v)}{\deg(\mathcal{H})} \\ &= \frac{b!((c-1)!)^b}{\binom{b}{d}(c-1)^d} \\ &= d!(b-d)!((c-1)!)^d((c-2)!)^{b-d}. \end{aligned}$$

And, by the above proof of 2.3 we know this is true! This is exactly one half of $\#\mathrm{Stab}_{S_c \wr S_b}\{v, x\}$, where because we are in $\mathrm{Stab}_{S_c \wr S_b}(v)$ we lose the option to send v to x and x to v by which we picked up the missing factor of two in the proof of the proposition. ♣

Remark 2.5. When we consider our results 1.10 and 2.3, we find that the labels $\varphi_1(\ell)$ and $\varphi_2(\ell)$ are well-defined for all $\ell \in \mathbb{N}$ in all weakly-general Hamming graphs. This is worth stating, as the remainder of our efforts will extensively focus on results concerning first and second labels.

Example 2.5.1 (No φ-labels for Triangles). Consider $\mathcal{H}_3^3(2)$. This graph contains the edge $\{(0,0,0),(0,1,1)\}$. Let's consider two maximal cliques built upon this edge. The first is a triangle, made with the addition of $(0,2,2)$. Let's write this out in matrix-like notation, where each row is home to a separate vertex and the columns are our indices. This notation has helped me to make speedy comparisons about distance, since that's really a function of counting disagreements column-by-column when you write this way.

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{matrix}$$

We can prove this is maximal. Take a string (x,y,z). The set $\{(0,0,0),(0,1,1),(0,2,2),(x,y,z)\}$ is not a 4-clique. To have $\boldsymbol{d}((0,0,0),(x,y,z)) = 2$, there must be two and exactly two nonzero entries. Our first case supposes $x = 0$. We have

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & y & z. \end{matrix}$$

This is already impossible! Since $y \neq 0$, it must be either 1 or 2, since those are our three options in this graph. Either way, it will agree with one of the two nonzero strings above it. Since the left column is uniformly zero, however, this is two agreements, where $d = 2$ forces exactly one agreement between any pair in a clique in this graph. If instead we took $y = 0$ (without loss of generality), we would have

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ x & 0 & z, \end{matrix}$$

which is still not any better for us. Since we already have our zero, we understand that $z \neq 0$ follows. This forces three disagreements with one of the nonzero strings we already had, and what we have is still not a clique.

Next, consider instead a different maximal clique containing that same edge we started with. I will write it in matrix-like notation for us.

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

One readily verifies this is a clique. Any subset of three vertices you consider is a triangle in this graph which is not maximal. As such, the idea of labels can only take us as far as edges.

2.2 The Global Clique Enumeration

The next theorem we will introduce is meant to turn the above theory into three suitably strong relations about the first and second ℓ-labels and the number of ℓ-cliques, $\#C_\ell$, in a given weakly-general Hamming graph. This explicitly relates the global problem of finding a term in the clique polynomial to the local problem of interrogating the structure of the subgraph $N(v)$, since this is where all cliques that contain some arbitrary vertex v or edge containing v will live.

Theorem 2.6 (Local-Global Translator)**.** *Let ℓ be a positive integer and $d \neq 0$. Then in the graph $\mathcal{H}_c^b(d)$ we have the relations*

$$\begin{aligned} i) \quad & \ell c_\ell = c^b \varphi_1(\ell); \\ ii) \quad & \binom{\ell}{2} c_\ell = |\mathcal{E}_c^b(d)| \varphi_2(\ell); \\ iii) \quad & \deg(\mathcal{H}) \varphi_2(\ell) = (\ell - 1)\varphi_1(\ell). \end{aligned}$$

Proof. Toward $i)$, we see for every vertex $v \in \mathbb{Z}_c^b$ that we get $\varphi_1(\ell)$ cliques in C_ℓ. But, each distinct clique will in this was be counted by first label once for each of ℓ vertices in it. Hence, the product $c^b \varphi_1(\ell)$ will be an overcount for c_ℓ by a factor of ℓ, and hence we recover the result $c_\ell = \frac{\varphi_1(\ell)}{\ell} c^b$, which is equivalent to $i)$.

The proof of $ii)$ is nearly identical in form and spirit. Every edge $e \in \mathcal{E}_c^b(d)$ will appear in precisely $\varphi_2(\ell)$ cliques in C_ℓ. As before, this produces an overcount. Each clique of order ℓ will be counted by second label for each of $\binom{\ell}{2}$ edges it contains. This establishes $c_\ell = \frac{\varphi_2(\ell)}{\binom{\ell}{2}} |\mathcal{E}_c^b(d)|$ and so proves $ii)$.

To finish the proof, one readily verifies $iii)$ follows from algebraic manipulations on $i)$ and $ii)$. ♣

Remark 2.7. Something like prong $ii)$ of this transliteration should be familiar to those versed in the property of arc-transitivity.

2.3 Local Counting Under Equivalence

Consider the subgraph of $\mathcal{H}_c^b(d)$ induced by $N(v) \subseteq \mathbb{Z}_c^b$. Theorem 2.6 indicates we should reduce the global clique enumeration problem to one which is local, such that we are able to maintain some handle on the structure which shows up there. The aim of this section is to introduce those structures on $N(v)$ which are in some sense very rich, especially when we consider the action of wreath product and its automorphisms.

First, we introduce an equivalence relation on skeletons comparing elements of $N(v)$ to v.

Definition 2.8 (Skeletal Equivalence)**.** For fixed $v \in \mathbb{Z}_c^b$, define a relation $\overset{\mathrm{Sk[v]}}{=\!=}$ on $N(v)$ by $x \overset{\mathrm{Sk[v]}}{=\!=} y$ iff $\mathrm{Sk}\{v, x\} = \mathrm{Sk}\{v, y\}$. The vertices x, y are said to be skeletal equivalent whenever this holds.

Definition 2.9 (Stack, Stack Skeleton)**.** Consider the equivalence class structure induced by the relation $\overset{\mathrm{Sk[v]}}{=\!=}$ on $N(v)$. We call an equivalence class $U \in N(v)/\overset{\mathrm{Sk[v]}}{=\!=}$ a stack of the induced subgraph $N(v)$, and we write $U \sqsubseteq N(v)$. We define $\mathrm{Sk}U = \mathrm{Sk}\{v, u\}$, the skeleton of the stack U, for any $u \in U \sqsubseteq N(v)$. Under the skeletal equivalence, this is clearly well-defined.

We now turn our attention to a more granular level, and study a class of objects which reside in the stack structure of $N(v)$.

Definition 2.10 (Matrix, Provenience)**.** Let $x \in X \sqsubseteq N(v)$. Consider a stack $U \sqsubseteq N(v)$. The matrix of X in $\mathrm{Sk}[U]$ is the set $M(X, U) = \mathrm{Sk}[X] \cap \mathrm{Sk}[U]$. We write $m(X, U) = \#M(X, U)$. Given $u \in U$ and the matrix $M(X, U)$, we define the provenience of u with respect to x by $P(x, u) = M(X, U) \cap \mathrm{Sk}\{x, u\}$. For its size, we write $p(x, u) = \#P(x, u)$.

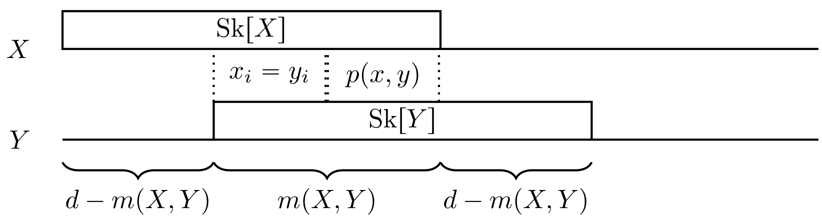
Think of the Matrix of two stacks as where they both necessitate disagreement with our string v. The provenience takes more specificity, calling upon the specific representatives of the stacks. It specifies where the elements disagree *within the matrix*. Outside of that, there are possible disagreements between x and u as a function of the fact that at most one of them can agree with v at indices outside of the matrix. This is a consequence of the fact the matrix is an intersection giving us exactly where they both disagree with v, according to their v-skeletons. A consequence of the appearance of indices at which at most one agrees with v outside their matrix is the following lemma.

Lemma 2.11 (Skeletal Decomposition Lemma)**.** *Let $x \in X \sqsubseteq N(v)$, $y \in Y \sqsubseteq N(v)$ in $\mathcal{H}_c^b(d)$. Then,* $\boldsymbol{d}(x, y) = 2d - 2m(X, Y) + p(x, y)$.

Proof. The set $\text{Sk}[X]\backslash M(X,Y)$ captures $d-m(X,Y)$ indices such that $y_i = v_i \neq x_i$. So, these contribute to the total disagreement between x and y. By a symmetry, we get another $d-m(X,Y)$ indices of disagreement from $\text{Sk}[Y]\backslash M(X,Y)$. Within $(\text{Sk}[X] \cup \text{Sk}[Y])^c$, it is clear that $x_i = v_i = y_i$.

Lastly, within $M(X,Y)$, we have defined a symbol $p(x,y)$ to count the number of disagreements between x and y here. Hence, the total disagreements are their sum, claimed quantity. ♣

Illustration. I hope to make this clear. Try to parse the pieces of the proof where they fit on this tikz picture. I have used the column permuting σ elements from $S_c \wr S_b$ to group labels together. A general clique still has the same counting structure in terms of $M(X,Y)$ and $p(x,y)$, but might not look so pretty and clustered.



Definition 2.12 (Substacks)**.** Let $U \sqsubseteq N[v]$. For any $x \in N(v)$, define the x-substack of U by $U|_{(x)} = \{y \in U : \boldsymbol{d}(x,y) = d\}$.

Theorem 2.13 (Matrix and Provenience Characterization)**.** *Let $x \in X \sqsubseteq N(v)$ and consider a stack $U \sqsubseteq N(v)$. The x-substack of U is given by $U|_{(x)} = \{u \in U : d - m(X,U) = m(X,U) - p(x,u)\}$.*

Proof. By the lemma, $\boldsymbol{d}\{x,u\} = 2d - 2m(X,U) + p(x,u)$, and one readily verifies $d = 2d - 2m(X,U) + p(x,u) \iff d - m(X,U) = P(x,u)$. ♣

Corollary 2.13.1 (Size of Substack)**.** *Given $x \in X \sqsubseteq N(v)$ and $U \sqsubseteq N(v)$, the size of $U|_{(x)}$ is precisely given by $\#U|_{(x)} = \binom{m(X,U)}{2m(X,U)-d}(c-1)^{d-m(X,U)}(c-2)^{2m(X,U)-d}$.*

Proof. We will count the choices of $u \in U$ for which $p(x,u) = 2m(X,U)-d$. Partition $\text{Sk}[U]$ into $\text{Sk}[U]\backslash\text{Sk}[X]$ and the matrix $M(X,U)$.

Within the former, each entry u_i of u has freedom to be any element of $\mathbb{Z}_c$ except v_i, and there are $d - m(X,U)$ indices here. So, we have $(c-1)^{d-m(X,U)}$ choices of substring within $\text{Sk}[U]\backslash\text{Sk}[X]$.

We must choose $P(x,u)$ within the matrix, and so there are $\binom{m(X,U)}{p(x,u)}$ possible proveniences. For each possible provenience, each $u \in U|_{(x)}$ will have for each entry u_i in its substring in $P(x,u)$ that $u_i \neq x_i$ and $u_i \neq v_i$, where because $P(x,u) \subseteq \text{Sk}[X]$ we note that $x_i \neq v_i$. Thus, at each of $p(x,u) = 2m(X,U) - d$ indices i the entry u_i can be any element of $\mathbb{Z}_c$ except for the two given by v_i and x_i.

Elsewhere in $M(X,U)\backslash P(x,u)$ the entries of u are uniquely determined by their agreement with corresponding entries of x. And, in $\text{Sk}[U]^c$, each entry of u is uniquely determined by agreement with v. Applying the product principle, we find that there are precisely $\binom{m(X,U)}{p(x,u)}(c-1)^{d-m(X,U)}(c-2)^{2m(X,U)-d}$ elements $u \in U|_{(x)}$. ♣

Example 2.13.1 (The coefficient c_3 of the clique polynomial for $\mathcal{H}_c^b(d)$). Note that, by Theorem 2.6, the number of triangles c_3 is a function of the number of triangles that contain a particular edge $\{v,x\}$, this quantity being the label $\varphi_2(3)$. For each element $u \in N(v)\backslash\{x\}$, note that $u \in U|_{(x)}$ if and only if $\{v,x,u\}$ is a triangle, writing $u \in U \sqsubseteq N(v)$.

By this observation, we see that every triangle that contains $\{v,x\}$ corresponds to an element in some substack. Note that $p(x,u)$ is nonnegative, and so a substack is nonempty and the above corollary applies only when $2m(X,U) - d$ is nonnegative. (We will see a generalization of this idea in Lemma 2.17.) So, we may address this question by counting the number of stacks corresponding to each natural $m \geqslant \frac{1}{2}d$, and write

$$\varphi_2(3) = \sum_{m \in [\frac{1}{2}d, d] \cap \mathbb{Z}} \#\{U \sqsubseteq N(v) : m(x,U) = m\} \binom{m}{2m-d} (c-1)^{d-m}(c-2)^{2m-d}.$$

To finish the enumeration, let us count via the product principle the number of stacks that induce a matrix of size m. Within Sk$[X]$, we must choose the matrix of size m. This can be done in one of $\binom{d}{m}$ ways. In Sk$[X]^C$, we have a $(b-d)$-set in which v and any $u \in U$ must agree in $b - 2d + m$ indices, since $\#\mathrm{Sk}[U] \backslash \mathrm{Sk}[X] = d - m$. Thus, U can take on $\binom{b-d}{d-m}$ possible sub-skeletons here.

We conclude

$$\varphi_2(3) = \sum_{m \in [\frac{1}{2}d, d] \cap \mathbb{Z}} \binom{b-d}{d-m} \binom{d}{m} \binom{m}{2m-d} (c-1)^{d-m}(c-2)^{2m-d},$$

and so one can verify algebraically from Theorem 2.6 that

$$c_3 = \frac{c^b(c-1)^d \binom{b}{d}}{6} \sum_{m \in [\frac{1}{2}d, d] \cap \mathbb{Z}} \binom{b-d}{d-m} \binom{d}{m} \binom{m}{2m-d} (c-1)^{d-m}(c-2)^{2m-d}.$$

2.4 Stack Homogeneity

Proposition 2.14 (Stack Repetition)**.** *Let $x, x' \in X \sqsubseteq N(v)$. Then, $x' \in X|_{(x)}$ if and only if $x_i \neq x'_i$ for all $i \in Sk[X]$.*

Proof. For $x' \in X$, we see $m(X,X) = d$ and so by proposition 2.11 we have $d = p(x', x)$ if and only if $\boldsymbol{d}(x', x) = d$. ♣

Definition 2.15 (Uniform Subclique, Stack Order)**.** For a clique $\chi \subseteq N(v)$ and a stack $U \sqsubseteq N(v)$, define its order with respect to χ, counting the number of elements in χ belonging to the stack U, by $\mathrm{ord}_\chi(U) = \#\chi \cap U$. If U has nonzero order, we define the uniform subclique χ_U by $\chi_U = \chi \cap U$.

Proposition 2.16 (Short Stack Agreement)**.** *Let $\{v, u, w\} \in C_3$, where $u \in U \sqsubseteq N(v)$ and $w \in W \sqsubseteq N(v)$. Then, $\#\{i \in Sk[U] : u_i = w_i\} = d - m(U,W)$.*

Proof. Clearly, $w \in W|_{(u)}$, and so the Theorem 2.13 implies that $p(u,w) = 2m(U,W) - d$. This is the number of indices i in a set of size $m(U,W)$ at which $u_i \neq w_i$. Thus, they agree at $m(U,W) - p(u,w) = d - m(U,W)$ indices there. ♣

Lemma 2.17 (Gravedigger's Lemma)**.** *Let χ be a clique with uniform subclique χ_U. If there exists $w \in W \sqsubseteq N(v)$ for which $\chi \cup \{w\}$ is a clique, then $m(W,U) \geqslant \frac{ord_\chi(U)}{1+ord_\chi(U)} d$.*

Proof. Write $n = \mathrm{ord}_\chi(U)$. Note that we have $m \geqslant \frac{n}{1+n} d \iff m \geqslant n(d-m)$. We will explain why the inequality in this form is true. Consider Sk$[U]$. We have Proposition 2.16 and so have for every $u \in \chi_U$ that there are $\#\{i \in \mathrm{Sk}[U] : u_i = w_i\} = d - m(U,W)$ indices at which u and w agree. See why any $i \in \mathrm{Sk}[U]$ admitting $u_i = w_i$ forces $i \in \mathrm{Sk}[W]$.

Now, by Proposition 2.14, at any index i for which $u_i = w_i$, then for any $u' \in \chi_U \backslash \{u\}$ we must have $u'_i \neq w_i$. This implies the sets $\{i \in M(W,U) : u_i = w_i\}$ induced by each $u \in \chi_U$ are disjoint. Note there is a unique such set for each u, of which there are n by assumption. So, we have accounted for n disjoint subsets of our matrix, each of size $d - m(W,U)$ indices. ♣

Corollary 2.17.1 (Homogeneity and Exhaustion)**.** *Suppose $\chi, \Upsilon \subseteq N(v)$ are cliques with $\chi \subseteq \Upsilon$ and $ord_\chi(U) \geqslant d$. If $W \sqsubseteq N(v)$, then $ord_\Upsilon(W) \neq 0$ if and only if $W = U$.*

Proof. Take $n > d - 1$ in the lemma. Then, $m(U,W) > d - 1$, where the fact that intersections of skeletons are subsets of a d-set tells us $m(U,W) = d$ is the only possibility. ♣

2.5 General Matrix-Provenience Analysis

Definition 2.18 (Generalized Matrix, Generalized Provenience)**.** Given a clique $\chi \subseteq N(v)$ and any stack $U \sqsubseteq N(v)$, define the matrix of U in χ by

$$M(U, \chi) = \mathrm{Sk}[U] \bigcap_{S:\, \chi_S \text{ exists}} \mathrm{Sk}[S].$$

For $u \in U$, define similarly the generalized provenience as

$$P(u, \chi) = M(U, \chi) \bigcap_{x \in \chi} \mathrm{Sk}\{x, u\}.$$

We use a lowercase $m(U, \chi)$ for the size of the generalized matrix and lowercase $p(u, \chi)$ for the size of the generalized provenience.

Definition 2.19 (Generalized Substacks)**.** Let $\chi \subseteq N(v)$ be a clique and let $U \sqsubseteq N(v)$. The χ-substack of U is defined $U|_{(\chi)} = \{u \in U : \forall x \in \chi, \boldsymbol{d}(u, x) = d\}$.

Proposition 2.20 (General Matrix & Provenience Characterization)**.** *Let $\chi \subseteq N(v)$ be a clique and let $x \in N(v) \backslash \chi$. Then, the set $\chi \cup \{x\}$ is a clique if and only if $p(x, \chi_U) = (ord_\chi(U) + 1)m(X, U) - ord_\chi(U)d$ for all stacks U with nonzero order in χ.*

Proof. Write $x \in X \sqsubseteq N(v)$ to refer to the stack to which x belongs. Let $U \sqsubseteq N(v)$ of order n and write $\chi_U = \{u_1, u_2, \dots, u_n\}$. Take any $u_i \in \chi_U$. Outside the matrix $M(X, U)$, we have seen in the proof of Lemma 2.11 that x and u_i disagree in $2(d - m(X, U))$ indices. Within $M(X, U)$, there are $d - m(X, U)$ indices at which x agrees with u_j for all $n-1$ elements $u_j \in \chi_U \backslash \{u_i\}$ by Proposition 2.16. All other disagreements must, by definition, lie within the generalized provenience. Thus, $\boldsymbol{d}(x, u_i) = 2(d - m(X, U)) + (n-1)(d - m(X, U)) + p(x, \chi_U)$.

Next, see that this implies $d = (n+1)(d - m(X, U)) + p(x, \chi_U)$ whenever $\boldsymbol{d}(x, u) = d$, which holds for all $u \in U$ whenever we assume $\chi_U \cup \{x\}$ is a clique and fails for at least one u_i in U whenever this set is not a clique. The remaining algebraic manipulations are straightforward. ♣

Corollary 2.20.1 (Size of Uniform Substacks)**.** *Take a clique $\chi = \chi_X$ for some $X \sqsubseteq N(v)$ and let $U \sqsubseteq N(v)$. Then, $\#U|_{(\chi)} = \binom{m(X,U)}{(ord_\chi(U)+1)m(X,U) - ord_\chi(U)d}(c-1)^{d-m(X,U)}(c - ord_\chi(X) - 1)^{(ord_\chi(X)+1)m(X,U) - ord_\chi(X)d}$.*

Proof. Write $n = \mathrm{ord}_\chi(X)$. We will count the choices of u in $U|_{(\chi)}$. At an index i outside the matrix $M(U, \chi)$, any $u \in U$ can disagree with $v_i = x_i$ (for all $x \in \chi$) via any of $c-1$ possible entries, and there are $d - m(U, \chi)$ indices with this property. Thus, for the substring of u outside the matrix, we have $(c-1)^{d-m(U,\chi)}$ choices. Within the complement of the provenience in the matrix, we have seen in Proposition 2.16 that u is always determined by agreement with v or some $x \in \chi$.

For reasons seen in Proposition 2.14 and by the very definition of provenience, we have at every index i within the provenience that u_i is forced to be any of the elements not in an $(n+1)$-subset of $\mathbb{Z}_c$. Since the proposition above states $p(u, \chi) = (n+1)m(U, \chi) - nd$, we have $(c - n - 1)^{(n+1)m(U,\chi) - nd}$ choices here. We must also do this for each possible provenience of this size. Taking the product, we get the claimed $\#U|_{(\chi)} = \binom{m(X,U)}{(\mathrm{ord}_\chi(U)+1)m(X,U) - \mathrm{ord}_\chi(U)d}(c-1)^{d-m(U,\chi)}(c-n-1)^{(n+1)m(U,\chi)-nd}$. ♣

3 Toward Counting by Composition

3.1 Substack Makeup of Cliques

Let χ be a clique with subcliques χ_U, χ_W, and suppose $u \in \chi_U$, $w \in \chi_W$. Then, $\#\{i \in M(U, W) : w_i = u_i\} = d - m(U, W)$ by Proposition 2.16. Proposition 2.20 also tells us that $p(w, \chi_U) = (\mathrm{ord}_\chi(U) + 1)m(U, W) - \mathrm{ord}_\chi(U)d$. This next lemma explains that we also have control over how much our chosen u, w disagree within the matrix but outside the provenience.

Lemma 3.1. *Let χ be a clique with subcliques χ_U, χ_W, and suppose $u \in \chi_U$, $w \in \chi_W$. Then, $\#\{i \in M(U,W)\backslash P(w,U) : w_i \neq u_i\} = (ord_\chi(U) - 1)(d - m(U,W))$.*

Proof. We have $\#\{i \in M(U,W) : w_i = u_i\} = d - m(U,W)$ and $p(w,\chi_U) = (\mathrm{ord}_\chi(U)+1)m(U,W) - \mathrm{ord}_\chi(U)d$. Note that we have a disjoint union

$$M(U,W) = \{i \in M(U,W) : w_i = u_i\} \sqcup \{i \in M(U,W)\backslash P(w,\chi_U) : w_i \neq u_i\} \sqcup P(w,\chi_U)$$

giving us an $m(U,W)$-set. In particular, the disjoint union of our matrix agreements and non-provenience disagreements between u, w is the whole of the complement of the provenience in the matrix. The claimed count arises naturally by disregarding $d - m(U,W)$ indices from the set of $m(U,W) - p(w,\chi_U) = \mathrm{ord}_\chi(U)(d - m(U,W))$ elements. ♣

Remark 3.2. For any clique χ with its subcliques χ_U, χ_W, we have a good degree of control of what the clique looks like. There is a sort of balance in its composition implied by this result. We know how big the provenience is by how many elements have been included from stacks U and W. We know how many agreements and pair of elements $u \in \chi_U$, $w \in \chi_W$ have in their matrix. We know how many disagreements are shared by this pair of representatives u, w where another representative agrees. In summary, we have a lot of control over what cliques look like at this point.

Theorem 3.3 (Gravedigger's Composition Theorem)**.** *For any subclique $\chi_U \subseteq N(v)$ and stack $W \subseteq N(v)$ such that $m(U,W) \geqslant \frac{ord_\chi(U)}{1+ord_\chi(U)}d$, the quantities $d - m(U,W)$, $(ord_\chi(U) - 1)(d - m(U,W)$, $(ord_\chi(U) + 1)m(U,W) - ord_\chi(U)d$ as given by the formulas above give a composition of $m(U,W)$.*

Proof. We verify that this composition exists for all M sufficiently large according to the Gravedigger's inequality 2.17 on the order of the stack U. It is clear that the formulas for the sizes of this representative triple of $M(U,W)$ sum to $m(U,W)$. What is left is to verify these sizes are nonnegative integers. We first note they are linear combinations of integers with integer coefficients. Since $m \leqslant d$, we see that $d - m$ and its nonnegative integer multiples are nonnegative. Also, see that $(n+1)m(U,W) - nd \geqslant 0$ if and only if the Gravedigger's inequality holds. ♣

Corollary 3.3.1 (Stack-Wise Maximality Test)**.** *If there exists an element w of W not already in χ such that this inequality holds for all stacks U with nonzero order in χ, then $\chi \cup \{w\}$ is a clique.*

Proof. See that, if this holds for a given stack U, it implies that $\boldsymbol{d}(w,u_j) = d$ for all $u_j \in \chi_U$. If this holds for each stack represented in χ, then it holds for all elements of χ. ♣

This is the point where a moral truth is taking shape. These graphs are all about three numbers. b tells you what skeletons you're going to have. d tells you how the skeletons interact in a given clique (given some of its internal organs). c tells you how many choices of a new appendage you have, once you've glued the skeletons together.

A Known Forms

Theorem A.1 (Isolated Graphs)**.** *Suppose $d = 0$. Note this includes the $c = 1$ case. $\mathcal{H}_c^b(0)$ has as its clique polynomial:* $1 + c^b t$

Proof. This graph consists of c^b isolated vertices. ♣

A.1 The Biggest and Smallest Skeletons

Theorem A.2 ($d = 1$)**.** *The unit-distance graph $\mathcal{H}_c^b(1)$ has as its clique polynomial:* $1 + c^b t + \frac{bc^b}{2}(c - 1) \sum_{j=2}^{c} \binom{c-2}{\ell-2} \frac{t^j}{\binom{\ell}{2}}$

Proof. First, we will count the number of ℓ-cliques that contain $\overline{0}$ and $(0, 0, \ldots, 0, c)$ to obtain the label $\varphi_2(\ell)$. Since $d = 1$, every clique is given by a uniform subclique corresponding to the stack whose skeleton consists only of the index b. We must choose $\ell - 2$ more entries here from $c - 2$ elements of $\mathbb{Z}_c$, excluding the zero and c in accordance with 2.14. So, $\varphi_2(\ell) = \binom{c-2}{\ell-2}$, and by theorem 2.6 the result follows. Note that a maximal clique in this graph is of size c. ♣

Theorem A.3 ($d = b$)**.** *The largest-distance graph $\mathcal{H}_c^b(b)$ has as its clique polynomial:* $1 + c^b t + \frac{c^b}{2}(c - 1)^b \sum_{j=2}^{c} \binom{c-2}{\ell-2} \frac{t^j}{\binom{\ell}{2}}$

Proof. First, we will count the number of ℓ-cliques that contain $\overline{0}$ and $(c, c, \ldots, c)$ to obtain the label $\varphi_2(\ell)$. Since $d = b$, there is one and only one stack. We must choose $\ell - 2$ more entries here from $c - 2$ elements of $\mathbb{Z}_c$, excluding the zero and c in accordance with 2.14. So, $\varphi_2(\ell) = \binom{c-2}{\ell-2}$, and by theorem 2.6 the result follows. Note that a maximal clique in this graph is of size c. ♣

Theorem A.4 ('Coming Soon': $c = 2$)**.** *The binary graph $\mathcal{H}_2^b(d)$ has as its clique polynomial:*

Proof. Since $c = 2$, proveniences must always be of size zero when comparing a candidate element to a nonempty uniform subclique. In particular, matrices must be of the least possible size to do this. ♣