1.4 Methods for Combinatorics

Actions are the words of heroes.
~ Dame Aylin

We have seen a theorem of Cayley which has, as its consequence, that every question about groups in some form expresses a question about permutations. The ultimate tool which takes this perspective is that of the *group action* on a set. These mappings are of particular interest to combinatorialists, and the bulk of this section will be our stating and proving theorems that get us to the heart of insights one makes by group counting. We present an example question that looks intractable absent a rigorous familiarity with so-called group actions.

Example 1.53. Consider the following illustration of four squares with their vertices colored red or black.

Fig. 1.14: Four 2-colored squares. Note D and B are twins under a 90° or 270° rotation, or horizontal or vertical reflection.

A natural curiosity of the combinatorialist presented with this setup is how many 2-colorings of the square there are. From the above example, there are at least three. We disregard the naive answer of $2^4 = 16$ as unfulfilling. Just as in squares D and B in this particular example, some of those options will be indistinguishable as far as the symmetries of the square are concerned.
What the combinatorialist means to ask is how many 2-colorings of the square are so meaningfully distinct that not even the dihedral symmetries of the square D_4 has the power to make them look alike. What about when we have 3 colors? 5? It is clear there should be a unique coloring if we only allow one color. Dependence on number of colors is also clear. For example, the number of colorings with three colors is at least three times that of two colors, as we can replicate all the possible two colored arrangements with every pair of colors from a g2iven set of three. Can we find a function for any positive integer n that counts the number of colorings up to dihedral symmetry? Spoiler alert: The answer is yes, $f(n) = \frac{1}{8}(n^4 + 2n^3 + 3n^2 + 2n)$, so in the case of the "2-colorings" of the square this function tells us there are six possibilities, what we will eventually call *orbits* of the action.

And now, we formally define the notion of a group action.

Definition 1.54 (Group Action)**.** Let G be a group and X a set. Denote by $\mathrm{Aut}(X)$ the group of all permutations of elements of X. That is, all bijective functions $X \to X$ under composition. Then, an *action of* G *on* X is a homomorphism of groups $\varphi : G \to \mathrm{Aut}(X)$. When such a φ exists, we write $G \overset{\varphi}{\curvearrowright} X$, omitting the homomorphism ϕ when it is canonical or clear.

Usually, we know what our action is in context so omit using its name as a homomorphism of groups φ, in which case we often coalesce the symbols g and $\varphi(g)$ into the group element g and the map $g : x \mapsto y$ on X if, for example $\varphi(g)(x) = y$. We will occasionally write $\varphi(g)(x)$ as a left translation action, i.e. if $\varphi(g)$ is an automorphism that happens sending x to y, we would write $g \cdot x = y$. There are also right translation actions and conjugation actions. We sometimes refer to the translation actions as "multiplication" actions because, in terms of the group, that is what they are.

Let us now unpack this definition more. A group action on a set is a homomorphism from the group and into the group of bijective functions on the set. In particular, it's a function sending elements of the group to permutations of the set such that this assignment is compatible with both the multiplication of the domain group G and the composition of permutations in $\mathrm{Aut}(X)$. We usually break this up in what gets called the *compatibility* of an action with the group.

There is an equivalent definition preferred by some that makes the compatibility front and center. Let G be a group and X a set. A function $f : G \to \mathrm{Aut}(X)$ is an action of G on X if and only if $f(1) = \mathrm{id}_X$ and $f(gh)(x) = f(g)(f(h)(x))$ for all $g, h \in G$, $x \in X$.

Exercise 1.55. Convince yourself that this definition is equivalent. Verify that this definition $f(gh)(x) = f(g)(f(h)(x))$ is the relevant homomorphism property in our first definition of the group action. See that, just as we proved in (refer to prior result), that identity is sent to identity follows from the homomorphism property that the multiplication is compatible.

There are a few quick examples of group actions we can inspect armed with this language.

Example 1.56. Consider the permutation group S_4. It has a natural action on $[4] = \{1, 2, 3, 4\}$, given it is isomorphic to $\mathrm{Aut}[4]$. As we have defined it, S_4 *is* the set of permutations of some set of four elements. This is actually why I prefer the homomorphism into an automorphism group picture of group actions. In scenarios like this one, it is much easier to use our knowledge of the structure and classification of groups to identify natural actions. There is another action on this set given by D_4. Let us use the plane-quadrants like notation and label the top right corner 1, the top left 2, the bottom left 3, and the bottom right 4. Then in the cycle notation $R_{90} \mapsto (1234)$ and $V \mapsto (12)(34)$.

Example 1.57. Consider the proof of Cayley's Theorem. We showed that a group permutes itself by left multiplication. But now we can understand that G defines an action on itself, given by the left multiplication. The particular map $G \to \mathrm{Aut}(G)$ is $g \mapsto (g : h \mapsto gh)$, sometimes called the *left-regular representation of* G. The proof of Cayley's Theorem verifies this is bijective. To check it is a homomorphism of G, we write $(gg')(h) = gg'h = g(g'h) = g(g'(h))$.

Next, we would like some language to discuss certain properties of group actions.

Definition 1.58 (Faithful Action)**.** Let $\varphi : G \to \mathrm{Aut}(X)$ be a group action. We say that G acts on X faithfully by φ when φ is injective. When the action is canonical or clear, we often just say that G acts faithfully on X.

Proposition 1.25 (Injectivity and Fixed Points)**.** *Suppose* $G \curvearrowright X$ *by* $\varphi : G \to \mathrm{Aut}(X)$. G *acts on* X *faithfully if and only if the only* $g \in G$ *such that* $g \cdot x = x$ *for all* $x \in X$ *is the identity* 1_G.

Proof. φ is injective if and only if has trivial kernel, such that the only group element sent to the identity map on X (the unique bijection with all points fixed) is 1. ♣

Example 1.59. The left-regular representation of a group is faithful. By uniqueness of the identity element, $gh = h$ for all h is only satisfied by $g = 1$. In fact, $gh = h$ for the particular h implies $g = 1$ so this holds for all h.

Example 1.60. The dihedral group D_3 acting on the vertices of an equilateral triangle in the usual manner is a faithful action. Only its identity, the zero degree rotation fixes all vertices. This example is not to say that every group action is faithful. In terms of D_3 on the set of vertices of an equilateral triangle, we may define the action by the trivial homomorphism sending everything to the identity permutation of the vertices. Clearly every element gives a map fixing each vertex when we force this picture.
We may also take the map to have as its kernel the rotation subgroup R_0, R_{120}, R_{240}. There are exactly two cosets in such case, and so to satisfy the First Isomorphism theorem the rotation subgroup must map to the identity map and then the remaining coset must map to some permutation of order two. Since we are dealing with permutations of a 3-set, there are exactly three choices. We ordinarily think of these transpositions as being given by the three reflections that are the representatives of this coset, and the background fact making it feel very natural is an isomorphism $D_3 \cong S_3$, but the fact we have nontrivial kernel and an action which is therefore not faithful should tell us to look more closely. We are treating the vertices of the triangle as our set X and D_3 a group which has some subgroup of $\text{Aut}(X)$ as a homomorphic image determining the action. It is within this homomorphism our action truly resides. In general, there are multiple homomorphisms $G \to \text{Aut}(X)$, and each gives a G-action on X in its own right.

Definition 1.61 (Free Action)**.** Let $\varphi : G \to \text{Aut}(X)$ be a group action. We say that $G \curvearrowright X$ freely under φ, and call φ a free action, if every $\varphi(g)$ for $g \neq 1_G$ has no fixed points. That is, if $g \neq 1$, then $g \cdot x \neq x$ is guaranteed for all $x \in X$.

Example 1.62. The left-regular representation is free as an action of a group G on itself. By cancellation laws, if $gh = h$ for any $h \in G$, then $g = 1$.

Example 1.63. The action of D_3 on the equilateral triangle is not free. The two nonidentity rotations have no fixed points, but each of three reflections fixes a vertex and transposes the other two.

Exercise 1.64. Convince yourself that being free is a stronger property of group actions than faithfulness. That is, verify every free action is faithful but not every faithful action is free.

Definition 1.65 (Transitive Action)**.** Let $\varphi : G \to \text{Aut}(X)$ be a group action. We say this action is transitive, or $G \curvearrowright X$ transitively under φ, if for every pair of elements $x, y \in X$ there exists $g \in G$ for which $\varphi(g)(x) = y$.

Example 1.66. In the left-regular representation of a group, we have transitivity. Let $x, y \in G$. To obtain $g \in G$ such that $gx = y$, we simply take $g = yx^{-1}$.

Example 1.67. The conjugation action of a nontrivial group on itself does not give a transitive action. In particular, $g1g^{-1} = 1$, so the identity only ever gets sent to itself under conjugation.

What you may wish to take from the above examples is that, in general, understanding an action will involve understanding where various elements of the set being acted on can land when acted upon by various group elements. Upon inspection, we should ascertain that there exists $g : x \mapsto y$ if and only if there exists $g^{-1} : y \mapsto x$. That is, inverse elements in the group allow us to walk back elements. Since groups have inverse elements, we get in particular that a given group action can send an element x to an element y if and only if it can send y to the element x. Moreover, by compatibility, if we have $h : x \mapsto y$ and $g : y \mapsto z$, then $gh \cdot x = g \cdot y = z$. And $1 \cdot x = x$.

Consequently, the action of G on X induces an equivalence relation. Say $x \sim y$ if and only if there exists $g : x \mapsto y$. The equivalence classes we partition X into based on the G-action have a very important name.

Definition 1.68 (Orbit)**.** Let $G \rightharpoonup X$. For any $x \in X$, we define its orbit under the action of G to be the set $\mathrm{Orb}_G(x) \overset{\text{def}}{=\!=} \{y \in X : x \sim y\}$. That is, the orbit of x is the set of elements x lands on under some permutation in the action of G. We denote the set of orbits by $X/G \overset{\text{def}}{=\!=} \{\mathrm{Orb}_G(x) : x \in X\}$.

Remark. In this language, we may say a group action is transitive if $\#X/G = 1$, i.e. there is a single orbit given by the entire set.

We verify the hinted equivalence relation occurs.

Proposition 1.26 (Set of Orbits is Partition)**.** *The set X/G of orbits of the action $G \rightharpoonup X$ is a partition of X.*

Proof. Let $y \in \mathrm{Orb}_G(x)$, writing $y = g \cdot x$. We will show $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$, where we are using implicitly $x \in \mathrm{Orb}_G(x)$ by $1 \in G$. Let $z \in \mathrm{Orb}_G(x)$. Say, $z = h \cdot x$. Then, $gh^{-1} \cdot z = gh^{-1}h \cdot x = g \cdot x = y$. Similarly, if $z = h \cdot y$, then $g^{-1}h^{-1} \cdot z = g^{-1} \cdot y = x$. ♣

Essentially, this proof follows from the fact that we have inverse elements and compatibility of the action. At the core of the proof is the fact that $y = g \cdot x \iff g^{-1} \cdot y = x$.

A common companion to this definition is the notion of the *stabilizer subgroup*. Together, they will soon come together in our first major theorem of this section.

Definition 1.69 (Stabilizer)**.** Let $G \rightharpoonup X$. For any $x \in X$, we define its stabilizer subgroup as $\mathrm{Stab}_G(x) \overset{\text{def}}{=\!=} \{g \in G : g \cdot x = x\}$. That is, the set group elements which stabilize x.

Remark. In this language, we may say an action is faithful if and only if the stabilizer subgroups intersect trivially and free if and only if all stabilizer subgroups are trivial.

Proposition 1.27 (Stabilizer Subgroup is Subgroup)**.** *Let $G \rightharpoonup X$ and $x \in X$. Then,* $\mathrm{Stab}_G(x) < G$.

Proof. It clearly contains 1, as $1 \cdot x = x$ is always true of a group action. Now, suppose $g, h \in \mathrm{Stab}_G(x)$. Then $g \cdot x = h \cdot x = x$, so furthermore we have $h^{-1}h \cdot x = h^{-1} \cdot x = x$. Thus, $gh^{-1} \cdot x = g \cdot (h^{-1} \cdot x) = g \cdot x = x$, so $gh^{-1} \in \mathrm{Stab}_G(x)$. ♣

Next, we prove a fundamental lemma connecting the orbit of an element and its stabilizer subgroup.

Lemma 1.28 (Cosets of Stabilizer Subgroup)**.** *Let $G \rightharpoonup X$ and $x \in X$. Then, the mapping*

$$\begin{aligned} f : \mathrm{Orb}_G(x) &\to G/\mathrm{Stab}_G(x) \\ g \cdot x &\mapsto g\mathrm{Stab}_G(x) \end{aligned}$$

is a bijection.

Proof. We will show it is invertible. Let $g\text{Stab}_G(x) = h\text{Stab}_G(x)$. Ours is the task to prove $g \cdot x = h \cdot x$ such that the obvious definition of an inverse map is well defined. To do so, consider $gh^{-1} \cdot x$. Since g, h are both representatives of the same coset of the stabilizer subgroup, then $gh^{-1} \in \text{Stab}_G(x)$ in which case $gh^{-1} \cdot x = x$. Then, by compatibility, $gh^{-1}h \cdot x = 1h \cdot x$, i.e. $g \cdot x = h \cdot x$. ♣

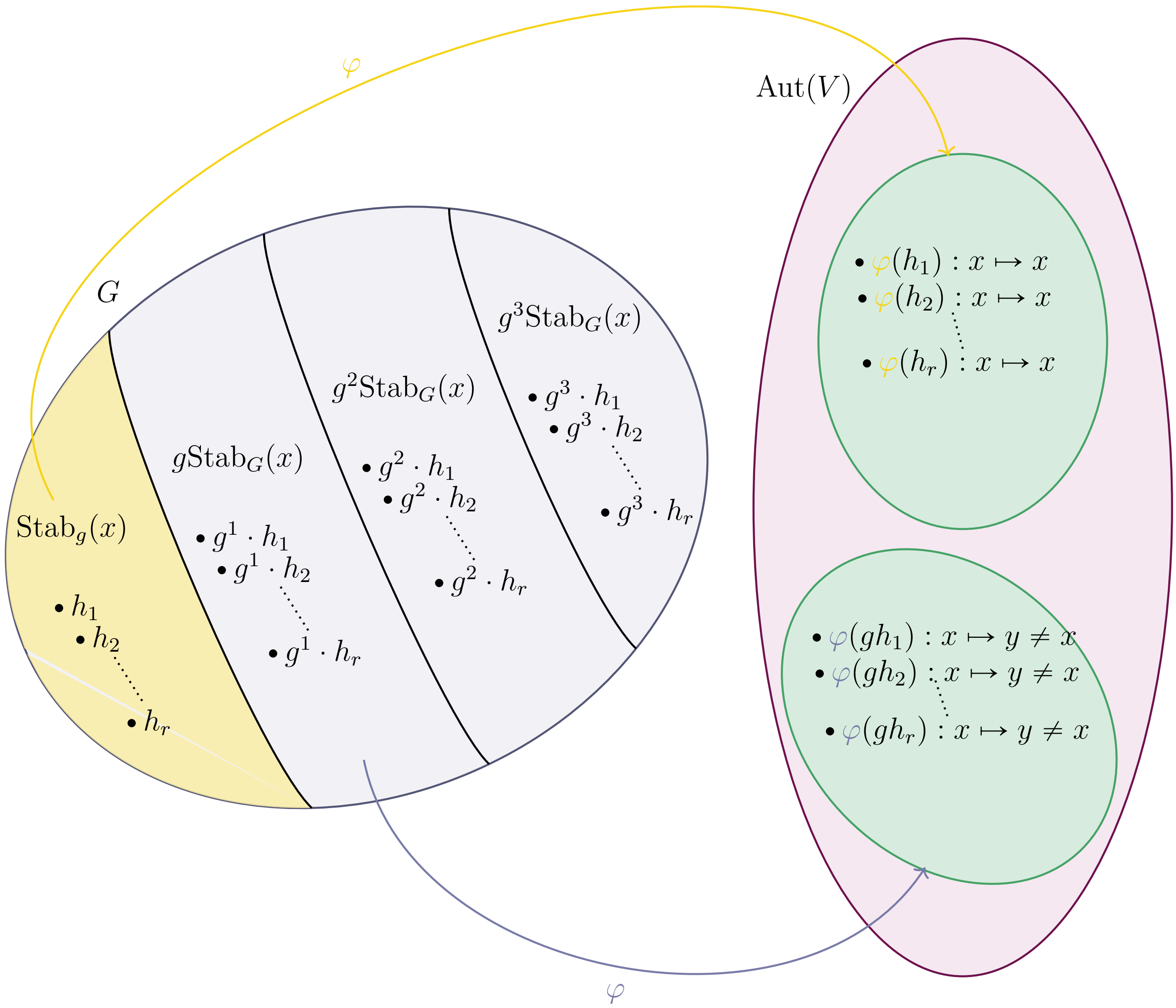

Fig. 1.15: Left cosets of the stabilizer subgroup and their action on $x \in X$ via $\varphi : G \to \text{Aut}(X)$. See why $gh_i \cdot x = g \cdot x$ if $h_i \cdot x = x$.

Example 1.70. We have discussed D_3 as a group of symmetries of the triangle and seen it is isomorphic to S_3 in our discussion of Cayley Tables. We will use this to illustrate the ways in which the action of D_3 on the set of vertices of an equilateral triangle are a homomorphism, and our picture will help us to see and visually trace this result on cosets of the stabilizer subgroup. In the standard case, the mapping is simply the isomorphism $D_3 \cong \text{Aut}V$ for vertex set V, so to make our drawing more interesting we consider just the rotational action whose kernel is the identity and re

We have sufficiently prepared ourselves for this next result such that it will likely not be so daunting. But it is so fundamental to so much of applied group theory, it deserves to be stated as our next major theorem.

Theorem 1.29 (Orbit-Stabilizer)**.** *Let $G \rightharpoonup X$ for G a finite group. For any $x \in X$, we have $\#G = \#\mathrm{Orb}_G(x)\#\mathrm{Stab}_G(x)$.*

Proof. By the lemma, we have $\#\mathrm{Orb}_G(x) = [G : \mathrm{Stab}_G(x)]$, and since the group is finite Lagrange's Theorem 1.10 applies and gives us the result. ♣

Example 1.71.

Exercise 1.72.

Next, we will take this discussion somewhere combinatorial by studying a first technique of group counting. It will take work on our part to make it useful and apply it to 2-colorings of the square and more. But we have already seen enough group theory to completely justify the methodology.

Lemma 1.30 (Reciprocal Partition Counting Trick)**.** *Suppose $G \rightharpoonup X$. Then, its set of orbits is counted by $\#X/G = \sum_{x \in X} \frac{\#\mathrm{Stab}_G(x)}{\#G}$.*

Proof. By the Orbit-Stabilizer Theorem,

$$\frac{1}{\#\mathrm{Orb}_G(x)} = \frac{\#\mathrm{Stab}_G(x)}{\#G}$$

holds for all $x \in X$. Take $A \in X/G$, and observe that

$$\sum_{a \in A} \frac{\#\mathrm{Stab}_G(a)}{\#G} = \frac{1}{\#\mathrm{Orb}_G(a)} = \#A\left(\frac{1}{\#A}\right) = 1,$$

since we took $A = \mathrm{Orb}_G(a)$ and the set of orbits gives a partition of X. This implies further

$$\sum_{x \in X} \frac{\#\mathrm{Stab}_G(x)}{\#G} = \sum_{A \in X/G} \sum_{a \in A} \frac{\#\mathrm{Stab}_G(a)}{\#G} = \sum_{A \in X/G} 1 = \#X/G.$$

♣

Remark. Though it adheres less neatly to our main story, the name of this lemma is given for the form $\#X/G = \sum_{x \in X} \frac{1}{\#\mathrm{Orb}_G(x)}$, the equivalence of which is an immediate consequence of the Orbit-Stabilizer theorem vital to the above proof. One notes we did not use any Group theory in the proof. Any partition of a finite set has this property and can be proved in the same way.

At this point, we have identified and orbit counting method. In practice, however, its current form is not what we want. This is because we have to determine quite a lot about each orbit to determine the reciprocals of orbit sizes or the stabilizer sizes themselves. If we were able to do all that, we could perform an orbit count that has nothing to do with the Orbit-Stabilizer Theorem. Let's do so explicitly to see how so.

Example 1.73. We will consider the set of vertices $V_T \overset{\text{def}}{=\!=} \{1, 2, 3, 4\}$ of the following regular tetrahedron.

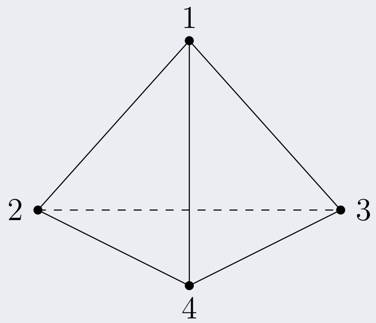

Fig. 1.16: Regular tetrahedron with vertices $1, 2, 3, 4 \in V_T$

Recall that

$$A_4 = \{\mathbb{I}, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}.$$

This is of order twelve as we expect. Observe that $A_4 \curvearrowright V_T$. Identity does the nothing we expect of it by homomorphism property. The three-cycles each serve to rotate a face around an axis through the face's midpoint through the opposing vertex (not in that face). We have studied $\mathbb{Z}/3\mathbb{Z}$ enough to note that this cyclic motion has two nonidentity elements who are one another's squares. There are four faces each, for the correct total of eight. Lastly, we need to account for three products of disjoint transpositions. Each one is realized as a 180° pivot of an orthogonal pair of axes. Convince yourself in your head that the rotation transposing any pair of vertices about the midpoint of their connecting edge also swaps the complementary pair. Really try to see this one!

We will consider the 2-colorings of this regular tetrahedron. We have already spoken informally of counting problems about colorings. Formally, for a positive integer k, a k-coloring of a polygon or polyhedron is a map from its set of vertices to a set of k colors, usually just the first k positive integers. Often, we will denote this something like $V_T^{[k]}$. This is in line with the fact that the number of functions possible is given by $(\#V_T)^k$.

For illustrative purposes, we will use blue and gold. Our goal is to build enough knowledge to use the reciprocal trick in determining the number of orbits of $A_4 \curvearrowright V_T$. Let x be the coloring $1 \mapsto \textit{blue}, 2 \mapsto \textit{gold}, 3 \mapsto \textit{gold}, 4 \mapsto \textit{blue}$. I will draw it for you.

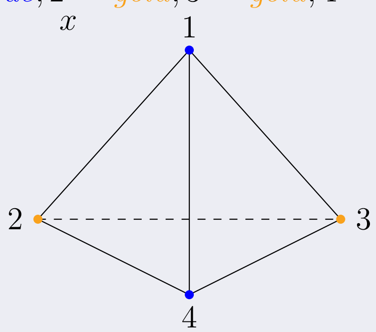

Fig. 1.17: Our regular tetrahedron with coloring x

We will now try to characterize $\mathrm{Orb}_{A_4}(x)$. I claim that this is the set of all $\binom{4}{2} = 6$ ways to color a pair of vertices blue and the other two gold. By three-cycles $(134), (143)$, we see that three of these six are covered immediately. But we also have the transposition $(12)(34)$. If we do this first, we may then see the other three as the result of those same $(134), (143)$. On the level of a counting argument, this gets us all six as follows. The vertex 2 is either blue or gold. If it is gold, a "two-gold-two-blue" arrangement requires exactly one of vertices $1, 3, 4$ be gold also. All three possibilities are achieved under the three-cycles we used, fixing 2 all the while. The (12)(34) element then makes 2 blue, and we repeat the process to move the other instance of a blue vertex to each of three possible positions. I will draw all six cases for you to check that the specific resultant element of A_4 does what I am saying.

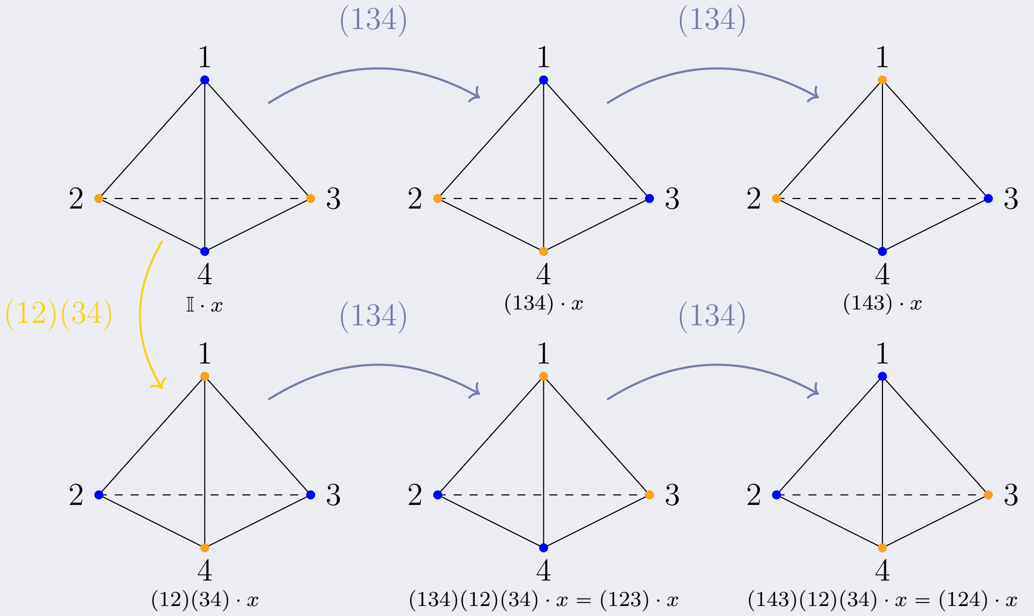

Fig. 1.18: The complete orbit of coloring x, including a representative from each left coset of $\text{Stab}_{A_4}(x)$, and paths showing what the products are doing as I constructed them to motivate this picture.

Next, we consider the coloring y given by $1 \mapsto$ *gold*, $2 \mapsto$ *blue*, $3 \mapsto$ *blue*, $4 \mapsto$ *blue*. There is a similar coloring we will call $\tilde{y}$ with the colors reversed, for three gold vertices and one blue.

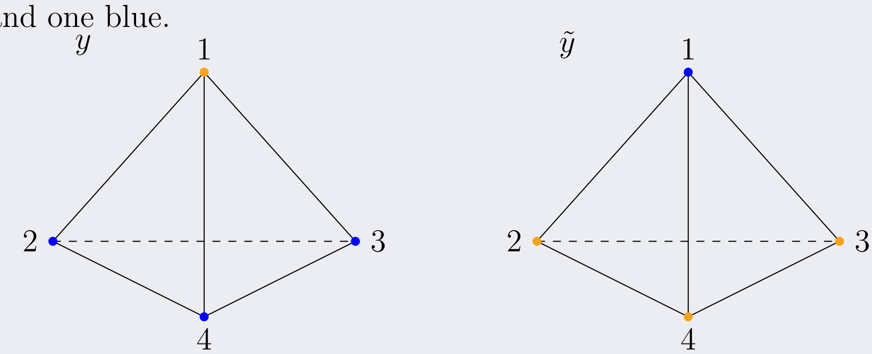

Fig. 1.19: Our regular tetrahedron with colorings $y, \tilde{y}$

We will count the orbit size again. Let us consider coloring y. See that we have a product of disjoint transpositions element that sends the gold vertex to any of the three other positions on what is the base, as I am drawing things. We can also think of this in terms of the stabilizer subgroup. A stabilizer must fix the gold 1. No products of disjoint transpositions do this. Three-cycles involving 1 fail to do this also. Thus, $\text{Stab}_{A_4}(y) = \{\mathbb{I}, (234), (243)\}$, in which case by Orbit-Stabilizer we confirm $\#\text{Orb}_{A_4}(y) = 4$. See why this is true of both y and $\tilde{y}$.

Finally, we consider the monochromatic colorings z, the all-blue coloring, and $\tilde{z}$, its all-gold counterpart. Observe that absolutely all of A_4 is in the stabilizer subgroup for either case. This means both orbits are singleton, since $\frac{\#A_4}{\#A_4} = 1$. Doing some accounting, there are sixteen maps from our four-set of vertices to the set of two colors we are considering. And, all have shown up. We have six "two-gold-two-blue" colorings making up $\mathrm{Orb}_{A_4}(x)$, eight from y and $\tilde{y}$, and two more given by z and $\tilde{z}$.

We have done the accounting for the orbit sizes for every element, and in doing so, we have considered five orbits. Namely, those of x, y, $\tilde{y}$, z, and $\tilde{z}$. But since we have exhausted everything to perform the reciprocal trick, we know these must be the five orbits, and so in context we can conclude the number of 2-colorings of a regular tetrahedron under symmetry is five. Be sure, the reciprocal trick still applies:

$$\sum_{c \in V_T^{[2]}} \frac{1}{\#\mathrm{Orb}_{A_4}(c)} = \underbrace{6\left(\frac{1}{6}\right)}_{x} + \underbrace{4\left(\frac{1}{4}\right)}_{y} + \underbrace{4\left(\frac{1}{4}\right)}_{\tilde{y}} + \underbrace{1\left(\frac{1}{1}\right)}_{z} + \underbrace{1\left(\frac{1}{1}\right)}_{\tilde{z}} = 5.$$

But, by the time we knew enough about our problem that we could use it, we no longer had need for it. We have to inspect each orbit, so have to identify a complete set of orbit representatives. In doing so, we merely look back at our work to find the number of orbits.

Toward a remedy of the problem in the above example, we will turn to one more definition fundamental to the mannerly combinatorialist's study of group actions.

Definition 1.74 (Fixed Point Set)**.** Let $G \curvearrowright X$. For any $g \in G$, we define its fixed point set to be the subset $\mathrm{Fix}_X(g) \stackrel{\text{def}}{=\!=} \{x \in X : g \cdot x = x\} \subset X$. That is, the fixed point set of g is the set of fixed points of the set automorphism associated to the particular action of g.

A brief historical note. Other books will refer to this next theorem as a Lemma of Burnside. Finish this historical note about how it was clear Cauchy knew this but didn't have modern language, and also Frobenius did prove it in more modern language......

Theorem 1.31 (Frobenius)**.** *Suppose $G \curvearrowright X$. Then, its set of orbits is counted by $\#X/G = \frac{1}{\#G}\sum_{g \in G} \mathrm{Fix}_X(g)$.*

Proof. This will follow from the reciprocal trick if we can show

$$\sum_{g \in G} \#\mathrm{Fix}_X(g) = \sum_{x \in X} \#\mathrm{Stab}_G(x),$$

and this can be accomplished with a combinatorial argument. Namely, we finish the proof by discussing how both summations count the number of pairs $(g, x) \in G \times X$ such that $g \cdot x = x$. That is, we convince ourselves

$$\sum_{g \in G} \#\mathrm{Fix}_X(g) = \#\{(g, x) : g \cdot x = x\} = \sum_{x \in X} \#\mathrm{Stab}_G(x),$$

to earn this theorem from the reciprocal trick by transitivity. The left hand side about fixed point sets says we should count this by considering the total number of pairs containing a particular $g \in G$, and then combine these as we vary g. This gets us back to fixed point sets in that the number of pairs (g^*, x) such that $g^* \cdot x = x$ with $g^* = g$ is given by the number of choices of x such that our $g^* = g$ fixes them, and so these $x \in X$ are precisely our fixed point set $\mathrm{Fix}_X(g^*)$. In symbols,

$$\#\{(g, x) : g \cdot x = x\} = \# \bigsqcup_{g^* \in G} \{(g^*, x) : g^* \cdot x = x\} = \sum_{g \in G} \#\mathrm{Fix}_X(g).$$

The right hand side about stabilizers says, analogously, we should consider the total number of pairs containing a particular $x \in X$, then combine these over all elements of X. Symbolically, this is nothing but

$$\#\{(g,x) : g \cdot x = x\} = \# \bigsqcup_{x^* \in X} \{(g,x^*) : g \cdot x^* = x^*\} = \sum_{x \in X} \#\mathrm{Stab}_G(x).$$

♣

Remark. Take a moment and reflect on the miracle, marvel, and spectacle of this result. We will address the practicalities and conveniences in a moment. The theorem itself says that the number of orbits is given by a sum of fixed point set sizes over all elements of the group, this expression divided by the order of the group. In particular, this expression on the right hand side is precisely the average size of a fixed point set out of the whole group. From the definitions alone and none of the theory we have developed, it is likely not obvious that the average size of a fixed point set be an integer. Not only is it now proven to be so, but it is a remarkably noteworthy integer in the context of the particular group action $G \curvearrowright X$. This theorem may have felt innocent, but it does say quite a lot about group actions and should make us much happier to accept them into our lives as algebraists, combinatorialists, or merely bright-spirited allies.

Now, let's try to conclude something worthwhile about an action of A_4 and colorings of the regular tetrahedron, to show off the power of this theorem in comparison with the reciprocal trick going purely by Orbit-Stabilizer.

Example 1.75. A_4 has twelve elements. They come in essentially three varieties: identity, 3-cycle, product of two disjoint transpositions. This view gets us in the direction of an important realization. What we are trying to do here is show that $\#V_T^{[2]}/A_4 = 5$, and incorporate fixed point sets as in the Frobenius theorem. Toward an understanding of fixed point sets, we need to know what it is about the pairing of a coloring of the vertices and some even permutation on the vertices that gives rise to stabilizing behavior.

Since the action of a particular element ultimately resolves into a *permutation* of a finite set, we can write it in terms of a product of disjoint cycles. We have even named our vertices such that this is easy to do, as the cycle notation there is exactly that with which we speak about A_4 to begin with. It is, in general, wise to do this when you can. Let us retrieve the two-gold-two-blue coloring we called x last time.

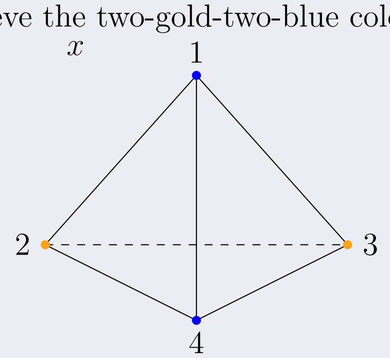

Fig. 1.20: Our regular tetrahedron with coloring x, to be considered anew, with better tools

I claim $(14)(23) \in \mathrm{Stab}_{A_4}(x)$, which can be checked by a quick inspection of the illustration. What we want is to explain why this is the case, and, in line with our prior findings, why it is the only nonidentity stabilizing element. I propose the following observation. Notice both cycles are monochromatic. That is, vertices 1 and 4 are both blue, and 2 and 3 are both gold. After a quick contemplation, this seems a reasonable criterion. It is necessary that cycles be monochromatic.

Else, you would (for example) be sending a blue vertex to a gold vertex, such that what was once gold ends up blue, and what you end up with does not look the same as what you started with because of it. And, it is sufficient. If every cycle is monochromatic, then every vertex in the cycle remains the same color before and after the action. Since all cycles look the same, all vertices must have been stable under the action.

So, then, we will use monochromatic cycles to count fixed point sets. Here is how. Every element of A_4 can be written as a product of disjoint cycles. For each cycle, we can choose a color, and our choice of one color per cycle induces an element of the fixed point set. So, the number of such choices counts the relevant fixed point set. I will quickly do this for A_4 and present it to you.

Element $\sigma \in A_4$	Cycle Type	Number of Cycles	$\#\text{Fix}_{V_T^{[2]}}(\sigma)$
$(1)(2)(3)(4)$	$(1,1,1,1)$	4	16
$(123)(4)$	$(3,1)$	2	4
$(132)(4)$	$(3,1)$	2	4
$(124)(3)$	$(3,1)$	2	4
$(142)(3)$	$(3,1)$	2	4
$(134)(2)$	$(3,1)$	2	4
$(143)(2)$	$(3,1)$	2	4
$(1)(234)$	$(3,1)$	2	4
$(1)(243)$	$(3,1)$	2	4
$(13)(24)$	$(2,2)$	2	4
$(14)(23)$	$(2,2)$	2	4
$(12)(34)$	$(2,2)$	2	4

Fig. 1.21: Table allowing us to organize the Frobenius theorem for the action of A_4 on the 2-colorings of a regular tetrahedron. Notice that, once we are acquainted with the method this table is built to help us keep track of, the only step that is not immediate is writing every element of the group in cycle notation.

Now, we apply the theorem.

$$\begin{aligned}\#V_T^{[2]}/A_4 &= \frac{1}{\#A_4}\sum_{\sigma\in A_4}\#\text{Fix}_{V_T^{[2]}}(\sigma)\\ &= \frac{1}{12}(16+11(4))\\ &= \frac{1}{12}(60)\\ &= 5,\end{aligned}$$

exactly what we had from before. But, notice how this time it was much easier and we actually used the proved counting method we set out to before we ever did so much mucking about that it became obsolete in context. Revel in the power of this tool. And dare to push it further with me. Nothing about monochromatic cycles or what they say about fixed point sets has anything in particular requiring we use a set of two colors. In general, the number of ways to color a fixed point of a group element with n disjoint cycles is k^n. Observe the last column above obeys this behavior. But this allows us to say something more general. Namely, we count the number of k-colorings of a regular tetrahedron by

$$\#V_T^{[k]}/A_4 = \frac{k^4+11k^2}{12}.$$

Exercise 1.76. In the hook for this section, we whizzed past the function $f(k) =$

$\frac{1}{8}(k^4 + 2k^3 + 3k^2 + 2k)$ which counts the number of k-colorings of the square. Using the action of D_4 on the square, construct a table as in the above example, and use it to show that this function is correct. Along the way, prove in abstract terms the various pieces we needed to motivate the table. For example, convince yourself in the abstract that for any such situation about colorings under a group action, a coloring is in the fixed point set of a group element if and only if the cycles of that group element are monochromatic under the coloring.